9709_31_Summer_2020_Q1

Solution

To solve the exponential inequality $2(3^{1-2x}) < 5^x$, we apply logarithmic properties to isolate the variable x.

1. Apply logarithms to both sides Since both sides of the inequality are positive, we can take the **natural logarithm** ($\ln$) of both sides. Because $\ln(x)$ is a strictly increasing function, the direction of the inequality remains unchanged:

$$\ln(2 \cdot 3^{1-2x}) < \ln(5^x)$$

2. Expand using logarithmic properties Using the **product rule** $\ln(ab) = \ln a + \ln b$ and the **power rule** $\ln(a^b) = b \ln a$:

$$\ln 2 + \ln(3^{1-2x}) < x \ln 5$$
$$\ln 2 + (1-2x)\ln 3 < x \ln 5$$

3. Isolate the variable x Distribute $\ln 3$ and group all terms containing x on one side:

$$\ln 2 + \ln 3 - 2x \ln 3 < x \ln 5$$
$$\ln 2 + \ln 3 < x \ln 5 + 2x \ln 3$$
$$\ln(2 \cdot 3) < x(\ln 5 + 2 \ln 3)$$
$$\ln 6 < x(\ln 5 + \ln 3^2)$$
$$\ln 6 < x(\ln 5 + \ln 9)$$
$$\ln 6 < x \ln(5 \cdot 9)$$
$$\ln 6 < x \ln 45$$

4. Solve for x Since $45 > 1$, $\ln 45$ is a positive value. Dividing both sides by $\ln 45$ preserves the inequality sign:

$$x > \frac{\ln 6}{\ln 45}$$

5. Boundary Case Verification Testing the boundary $x = \frac{\ln 6}{\ln 45}$:

$$2(3^{1-2\frac{\ln 6}{\ln 45}}) = 2 \cdot 3 \cdot 3^{-2\frac{\ln 6}{\ln 45}} = 6 \cdot (3^{\log_3 45})^{-\frac{\ln 6}{\ln 45}}$$

$$\text{Using base change: } \frac{\ln 6}{\ln 45} = \log_{45} 6$$

$$6 \cdot 3^{-2\log_{45} 6} = 6 \cdot (3^2)^{-\log_{45} 6} = 6 \cdot 9^{-\log_{45} 6}$$

$$\text{At the boundary: } 2(3^{1-2x}) = 5^x \implies \ln 6 = x \ln 45 \implies x = \frac{\ln 6}{\ln 45}$$

The boundary condition holds exactly. Since the original inequality was strictly “less than”, and the coefficient of x in the final step was positive, the solution set is all values greater than the critical point.

$$\boxed{x > \frac{\ln 6}{\ln 45}}$$

9709_31_Summer_2020_Q2

Solution

1. Binomial Expansion of $(2-3x)^{-2}$

To expand $(2-3x)^{-2}$ using the **Binomial Theorem** for a general power, we first rewrite the expression in the standard form $(1+u)^n$ by factoring out the constant term.

- **Step 1: Factor out the constant**

$$\begin{aligned}(2-3x)^{-2} &= \left[2\left(1-\frac{3}{2}x\right)\right]^{-2} \\ &= 2^{-2}\left(1-\frac{3}{2}x\right)^{-2} \\ &= \frac{1}{4}\left(1-\frac{3}{2}x\right)^{-2}\end{aligned}$$

- **Step 2: Apply the Binomial Series formula** The general expansion for $(1+u)^n$ is given by:

$$(1+u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \ldots$$

  Substituting $n=-2$ and $u=-\frac{3}{2}x$:

$$\begin{aligned}\left(1-\frac{3}{2}x\right)^{-2} &= 1 + (-2)\left(-\frac{3}{2}x\right) + \frac{(-2)(-3)}{2}\left(-\frac{3}{2}x\right)^2 + \ldots \\ &= 1 + 3x + 3\left(\frac{9}{4}x^2\right) + \ldots \\ &= 1 + 3x + \frac{27}{4}x^2 + \ldots\end{aligned}$$

- **Step 3: Multiply by the external factor**

$$\begin{aligned}(2-3x)^{-2} &= \frac{1}{4}\left(1 + 3x + \frac{27}{4}x^2 + \ldots\right) \\ &= \frac{1}{4} + \frac{3}{4}x + \frac{27}{16}x^2 + \ldots\end{aligned}$$

2. Validity of the Expansion

The **radius of convergence** for the binomial series $(1+u)^n$ requires that $\mid u \mid < 1$. In this specific case, the variable part of the binomial is $u = -\frac{3}{2}x$.

- **Step 1: Set up the inequality**

$$\left|-\frac{3}{2}x\right| < 1$$

- **Step 2: Solve for x**

$$\frac{3}{2} \mid x \mid < 1$$

$$\mid x \mid < \frac{2}{3}$$

This can also be expressed as $-\frac{2}{3} < x < \frac{2}{3}$.

Final Answer:

(a) $\frac{1}{4} + \frac{3}{4}x + \frac{27}{16}x^2$

(b) $\mid x \mid < \frac{2}{3}$

9709_31_Summer_2020_Q3

Solution

1. Expansion of Trigonometric Terms

To express the equation $\tan(\theta + 60^\circ) = 2 + \tan(60^\circ - \theta)$ in terms of $\tan\theta$, we apply the **tangent addition and subtraction formulas**:

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Given $\tan 60^\circ = \sqrt{3}$, we substitute this into the expansions:

- $\tan(\theta + 60^\circ) = \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}$
- $\tan(60^\circ - \theta) = \frac{\sqrt{3} - \tan\theta}{1 + \sqrt{3}\tan\theta}$

Substituting these into the original equation:

$$\frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} = 2 + \frac{\sqrt{3} - \tan\theta}{1 + \sqrt{3}\tan\theta}$$

2. Derivation of the Quadratic Equation

Let $x = \tan\theta$. The equation becomes:

$$\frac{x + \sqrt{3}}{1 - \sqrt{3}x} - \frac{\sqrt{3} - x}{1 + \sqrt{3}x} = 2$$

To simplify, find a common denominator for the left-hand side:

$$\frac{(x + \sqrt{3})(1 + \sqrt{3}x) - (\sqrt{3} - x)(1 - \sqrt{3}x)}{(1 - \sqrt{3}x)(1 + \sqrt{3}x)} = 2$$

$$\frac{(x + \sqrt{3}x^2 + \sqrt{3} + 3x) - (\sqrt{3} - 3x - x + \sqrt{3}x^2)}{1 - 3x^2} = 2$$

$$\frac{(\sqrt{3}x^2 + 4x + \sqrt{3}) - (\sqrt{3}x^2 - 4x + \sqrt{3})}{1 - 3x^2} = 2$$

$$\frac{8x}{1 - 3x^2} = 2$$

Dividing both sides by 2 and rearranging:

$$\frac{4x}{1 - 3x^2} = 1$$

$$4x = 1 - 3x^2$$

$$3x^2 + 4x - 1 = 0$$

Thus, the quadratic equation in $\tan\theta$ is:

$$\boxed{3\tan^2\theta + 4\tan\theta - 1 = 0}$$

3. Solving for θ

We solve the quadratic equation $3x^2 + 4x - 1 = 0$ using the **quadratic formula**:

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4(3)(-1)}}{2(3)} \\ &= \frac{-4 \pm \sqrt{16+12}}{6} \\ &= \frac{-4 \pm \sqrt{28}}{6} = \frac{-4 \pm 2\sqrt{7}}{6} = \frac{-2 \pm \sqrt{7}}{3} \end{aligned}$$

Calculating the numerical values for $\tan\theta$:

- $\tan\theta_1 = \frac{-2+\sqrt{7}}{3} \approx 0.21525$
- $\tan\theta_2 = \frac{-2-\sqrt{7}}{3} \approx -1.54858$

Now, we find the values of θ in the interval $0^\circ \le \theta \le 180^\circ$:

- For $\tan\theta_1 \approx 0.21525$:

$$\theta_1 = \arctan(0.21525) \approx 12.1448^\circ$$

- For $\tan\theta_2 \approx -1.54858$: Since the tangent is negative, θ must be in the second quadrant ($90^\circ < \theta < 180^\circ$):

$$\theta_2 = 180^\circ + \arctan(-1.54858) \approx 180^\circ - 57.1448^\circ = 122.8552^\circ$$

Rounding to one decimal place:

$$\boxed{\theta = 12.1^\circ, 122.9^\circ}$$

9709_31_Summer_2020_Q4

Solution

To find the stationary point of the curve $y = e^{2x}(\sin x + 3\cos x)$ in the interval $0 \le x \le \pi$, we follow the principles of **differential calculus**.

1. Finding the first derivative A **stationary point** occurs where the first derivative dy/dx is equal to zero. We apply the **product rule**, which states that for $y = uv$, $dy/dx = u'v + uv'$. Let $u = e^{2x}$ and $v = \sin x + 3\cos x$.

- $u' = 2e^{2x}$
- $v' = \cos x - 3\sin x$

The derivative is:

$$\begin{aligned}\frac{dy}{dx} &= 2e^{2x}(\sin x + 3\cos x) + e^{2x}(\cos x - 3\sin x) \\ &= e^{2x}[2\sin x + 6\cos x + \cos x - 3\sin x] \\ &= e^{2x}(7\cos x - \sin x)\end{aligned}$$

2. Solving for the x-coordinate Set $dy/dx = 0$:

$$e^{2x}(7\cos x - \sin x) = 0$$

Since $e^{2x} \neq 0$ for all real x, we must have:

$$\begin{aligned}7\cos x - \sin x &= 0 \\ \sin x &= 7\cos x \\ \tan x &= 7\end{aligned}$$

We solve for x in the interval $0 \le x \le \pi$:

$$x = \arctan(7) \approx 1.428899...$$

Rounding to 2 decimal places, we obtain $x \approx 1.43$.

3. Determining the nature of the stationary point To determine if the point is a maximum or minimum, we evaluate the **second derivative** d^2y/dx^2 at $x = \arctan(7)$. Starting from $dy/dx = e^{2x}(7\cos x - \sin x)$:

$$\begin{aligned}\frac{d^2y}{dx^2} &= 2e^{2x}(7\cos x - \sin x) + e^{2x}(-7\sin x - \cos x) \\ &= e^{2x}[14\cos x - 2\sin x - 7\sin x - \cos x] \\ &= e^{2x}(13\cos x - 9\sin x)\end{aligned}$$

At the stationary point, we know $\sin x = 7\cos x$. Substituting this into the second derivative expression:

$$\begin{aligned}\frac{d^2y}{dx^2} &= e^{2x}(13\cos x - 9(7\cos x)) \\ &= e^{2x}(13\cos x - 63\cos x) \\ &= -50e^{2x}\cos x\end{aligned}$$

Since $x = \arctan(7)$ is in the first quadrant ($0 < x < \pi/2$), $\cos x > 0$. Also, $e^{2x} > 0$. Therefore, $d^2y/dx^2 = -50e^{2x}\cos x < 0$. According to the **second derivative test**, a negative second derivative indicates a **local maximum**.

[Visualization]

(a) The x-coordinate is $\boxed{1.43}$. **(b)** The stationary point is a $\boxed{\text{maximum}}$.

9709_31_Summer_2020_Q5

Solution

1. Polynomial Division

To find the quotient and remainder of the division of $P(x) = 2x^3 - x^2 + 6x + 3$ by $D(x) = x^2 + 3$, we use **polynomial long division**.

- **Step 1:** Divide the leading term of the dividend ($2x^3$) by the leading term of the divisor (x^2).

$$\frac{2x^3}{x^2} = 2x$$

  This is the first term of the quotient.

- **Step 2:** Multiply the divisor by $2x$ and subtract from the dividend.

$$(2x^3 - x^2 + 6x + 3) - 2x(x^2 + 3) = (2x^3 - x^2 + 6x + 3) - (2x^3 + 6x) = -x^2 + 3$$

- **Step 3:** Divide the leading term of the new polynomial ($-x^2$) by the leading term of the divisor (x^2).

$$\frac{-x^2}{x^2} = -1$$

  This is the second term of the quotient.

- **Step 4:** Multiply the divisor by -1 and subtract.

$$(-x^2 + 3) - (-1)(x^2 + 3) = (-x^2 + 3) - (-x^2 - 3) = 6$$

The quotient is $2x - 1$ and the remainder is 6. Thus, we can write:

$$\frac{2x^3 - x^2 + 6x + 3}{x^2 + 3} = 2x - 1 + \frac{6}{x^2 + 3}$$

$$\boxed{\text{Quotient: } 2x - 1, \text{ Remainder: } 6}$$

2. Definite Integral

Using the result from part (a), we evaluate the **definite integral**:

$$I = \int_1^3 \left(2x - 1 + \frac{6}{x^2 + 3}\right) dx$$

- **Step 1:** Split the integral into basic components.

$$I = \int_1^3 (2x - 1) dx + \int_1^3 \frac{6}{x^2 + 3} dx$$

- **Step 2:** Evaluate the polynomial part.

$$\int_1^3 (2x-1)dx = [x^2 - x]_1^3$$
$$= (3^2 - 3) - (1^2 - 1)$$
$$= (9 - 3) - (0)$$
$$= 6$$

- **Step 3:** Evaluate the rational part using the **standard integral** $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan(\frac{x}{a}) + C$. Here, $a^2 = 3$, so $a = \sqrt{3}$.

$$\int_1^3 \frac{6}{x^2+3}dx = 6\left[\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_1^3$$
$$= \frac{6}{\sqrt{3}}\left(\arctan\left(\frac{3}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right)$$
$$= 2\sqrt{3}\left(\arctan(\sqrt{3}) - \arctan\left(\frac{\sqrt{3}}{3}\right)\right)$$

- **Step 4:** Substitute the known values for the inverse trigonometric functions.

$$\arctan(\sqrt{3}) = \frac{\pi}{3}$$
$$\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

$$2\sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 2\sqrt{3}\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}\pi}{3}$$

- **Step 5:** Combine the results.

$$I = 6 + \frac{\sqrt{3}\pi}{3}$$

$$\boxed{6 + \frac{\sqrt{3}\pi}{3}}$$

9709_31_Summer_2020_Q6

Solution

1. Derivation of the equation for x

- The shaded region is bounded by the tangents AT and BT and the minor arc AB.
- Since AT and BT are tangents to the circle at A and B, the angles $\angle OAT$ and $\angle OBT$ are right angles (90° or $\pi/2\,\text{rad}$).
- The line OT bisects the angle $\angle AOB$. Therefore, in the right-angled triangle $\triangle OAT$:

$$\angle AOT = \frac{1}{2}\angle AOB = \frac{1}{2}(2x) = x\,\text{rad}$$

- Using trigonometry in $\triangle OAT$:

$$\tan x = \frac{AT}{OA} = \frac{AT}{r} \implies AT = r\tan x$$

- The area of the quadrilateral $OATB$ is the sum of the areas of two congruent triangles, $\triangle OAT$ and $\triangle OBT$:

$$\text{Area}_{OATB} = 2 \times \left(\frac{1}{2} \times OA \times AT\right) = r \times (r\tan x) = r^2 \tan x$$

- The area of the **sector** OAB is given by:

$$\text{Area}_{\text{sector}} = \frac{1}{2}r^2(2x) = r^2 x$$

- The area of the shaded region is the difference between the area of the quadrilateral and the area of the sector:

$$\text{Area}_{\text{shaded}} = r^2\tan x - r^2 x = r^2(\tan x - x)$$

- Given that the area of the shaded region is equal to the area of the circle (πr^2):

$$r^2(\tan x - x) = \pi r^2$$

$$\tan x - x = \pi$$

$$\tan x = \pi + x$$

2. Verification of the root interval

- Let $f(x) = \tan x - x - \pi$. We check for a sign change between $x = 1$ and $x = 1.4$.
- For $x = 1$:

$$f(1) = \tan(1) - 1 - \pi \approx 1.5574 - 1 - 3.1416 = -2.5842$$

- For $x = 1.4$:

$$f(1.4) = \tan(1.4) - 1.4 - \pi \approx 5.7979 - 1.4 - 3.1416 = 1.2563$$

- Since $f(1) < 0$ and $f(1.4) > 0$, and $f(x)$ is continuous on the interval $[1, 1.4]$ (as $1.4 < \pi/2$), by the **Intermediate Value Theorem**, there must be at least one root in the interval $(1, 1.4)$.

3. Iterative solution

- We use the **iterative formula** $x_{n+1} = \tan^{-1}(\pi + x_n)$. We choose a starting value within the interval, for example, $x_0 = 1.2$.
- **Iteration 1**:

$$x_1 = \tan^{-1}(\pi + 1.2) = \tan^{-1}(4.34159...) \approx 1.3443$$

- **Iteration 2**:

$$x_2 = \tan^{-1}(\pi + 1.3443) = \tan^{-1}(4.48589...) \approx 1.3514$$

- **Iteration 3**:

$$x_3 = \tan^{-1}(\pi + 1.3514) = \tan^{-1}(4.49299...) \approx 1.3518$$

- **Iteration 4**:

$$x_4 = \tan^{-1}(\pi + 1.3518) = \tan^{-1}(4.49339...) \approx 1.3518$$

The values have converged to four decimal places. Rounding to two decimal places, we get $x \approx 1.35$.

$\boxed{x = 1.35}$

9709_31_Summer_2020_Q7

Solution

1. Differentiation of the function

To find the derivative of $f(x) = \frac{\cos x}{1+\sin x}$, we apply the **quotient rule**:

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

- Let $u(x) = \cos x$, then $u'(x) = -\sin x$.
- Let $v(x) = 1 + \sin x$, then $v'(x) = \cos x$.

Substituting these into the quotient rule formula:

$$\begin{aligned} f'(x) &= \frac{(-\sin x)(1+\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} \\ &= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} \end{aligned}$$

Using the **Pythagorean identity** $\sin^2 x + \cos^2 x = 1$:

$$\begin{aligned} f'(x) &= \frac{-\sin x - 1}{(1+\sin x)^2} \\ &= \frac{-(1+\sin x)}{(1+\sin x)^2} \\ &= -\frac{1}{1+\sin x} \end{aligned}$$

Analysis of the sign of $f'(x)$:

- For the interval $-\frac{1}{2}\pi < x < \frac{3}{2}\pi$, the value of $\sin x$ ranges from $-1 < \sin x \leq 1$.
- Specifically, $\sin x = -1$ only at $x = -\frac{1}{2}\pi$ and $x = \frac{3}{2}\pi$, which are the boundaries (not included in the open interval).
- Therefore, for all x in $(-\frac{1}{2}\pi, \frac{3}{2}\pi)$, we have $\sin x > -1$.
- This implies $1 + \sin x > 0$.
- Since the denominator is positive and the numerator is -1, $f'(x) < 0$ for all x in the given interval.

2. Integration of the function

To find the **definite integral** $I = \int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \frac{\cos x}{1+\sin x} dx$, we use the method of **u-substitution**.

- Let $u = 1 + \sin x$.
- Then $du = \cos x \, dx$.

Change of limits:

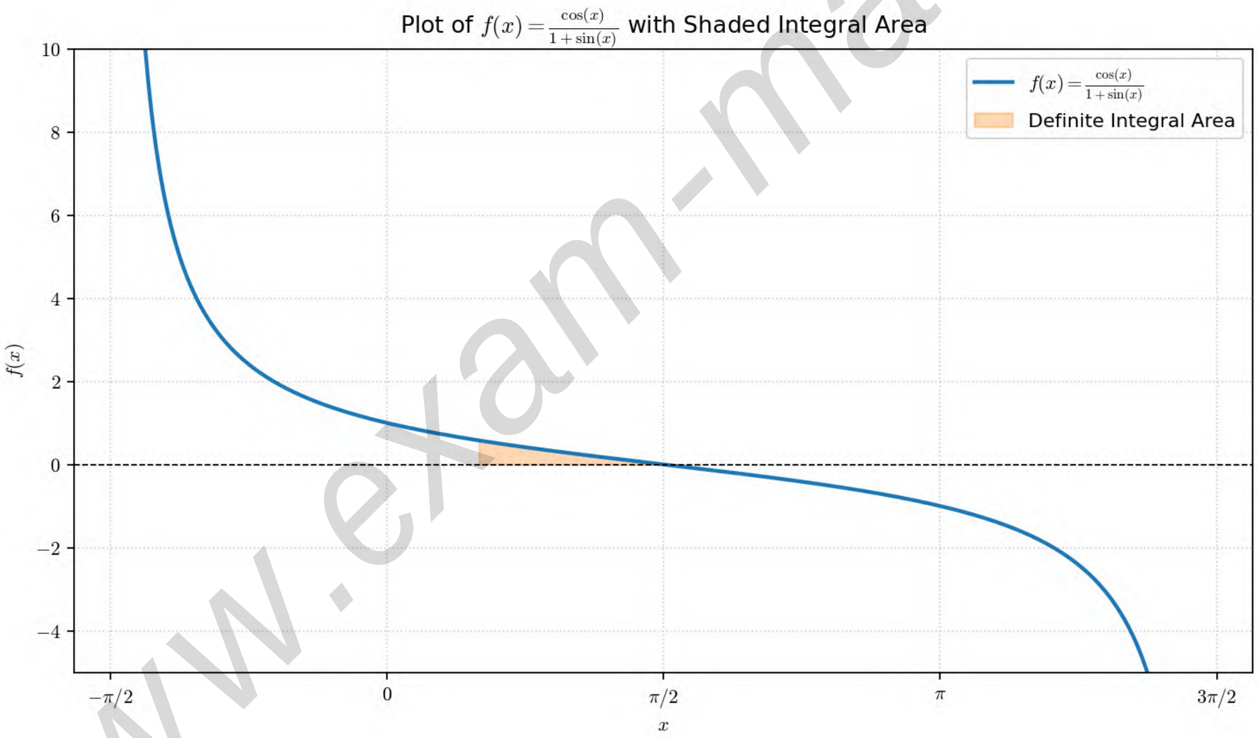
- When $x = \frac{1}{6}\pi$, $u = 1 + \sin(\frac{1}{6}\pi) = 1 + \frac{1}{2} = \frac{3}{2}$.
- When $x = \frac{1}{2}\pi$, $u = 1 + \sin(\frac{1}{2}\pi) = 1 + 1 = 2$.

Substituting into the integral:

$$I = \int_{3/2}^{2} \frac{1}{u} du$$
$$= [\ln|\ u\ |]_{3/2}^{2}$$
$$= \ln(2) - \ln\left(\frac{3}{2}\right)$$

Using the **logarithm power rule** $\ln a - \ln b = \ln(\frac{a}{b})$:

$$I = \ln\left(\frac{2}{3/2}\right)$$
$$= \ln\left(\frac{4}{3}\right)$$



The exact value of the integral is:

$$\boxed{\ln\left(\frac{4}{3}\right)}$$

9709_31_Summer_2020_Q8

Solution

1. Formulation of the Differential Equation

The problem states that the **gradient** of the curve at any point (x, y) is proportional to $\frac{y}{x\sqrt{x}}$. In calculus, the gradient is represented by the derivative $\frac{dy}{dx}$. We can express this relationship as:

$$\frac{dy}{dx} = k\frac{y}{x\sqrt{x}}$$

where k is a constant of proportionality. Since $x\sqrt{x} = x^{3/2}$, the equation becomes:

$$\frac{dy}{dx} = k\frac{y}{x^{3/2}}$$

2. Solving the Differential Equation

To solve this, we use the method of **separation of variables**:

- Rearrange the terms to group y on the left and x on the right:

$$\frac{1}{y}dy = kx^{-3/2}dx$$

- Integrate both sides:

$$\int \frac{1}{y}dy = \int kx^{-3/2}dx$$

- Applying the **power rule** for integration:

$$\ln|\ y\ | = k\frac{x^{-1/2}}{-1/2} + C$$

$$\ln|\ y\ | = -2kx^{-1/2} + C$$

- For simplicity, let $A = -2k$. The equation is:

$$\ln|\ y\ | = \frac{A}{\sqrt{x}} + C$$

3. Determining the Constants

The curve passes through the points $(1, 1)$ and $(4, e)$. We use these **boundary conditions** to find A and C.

- **Using point $(1, 1)$:**

$$\ln(1) = \frac{A}{\sqrt{1}} + C$$
$$0 = A + C \implies C = -A$$

- **Using point $(4, e)$:**

$$\ln(e) = \frac{A}{\sqrt{4}} + C$$

$$1 = \frac{A}{2} + C$$

- Substitute $C = -A$ into the second equation:

$$1 = \frac{A}{2} - A$$

$$1 = -\frac{A}{2}$$

$$A = -2$$

- Consequently, $C = -(-2) = 2$.

4. Final Equation of the Curve

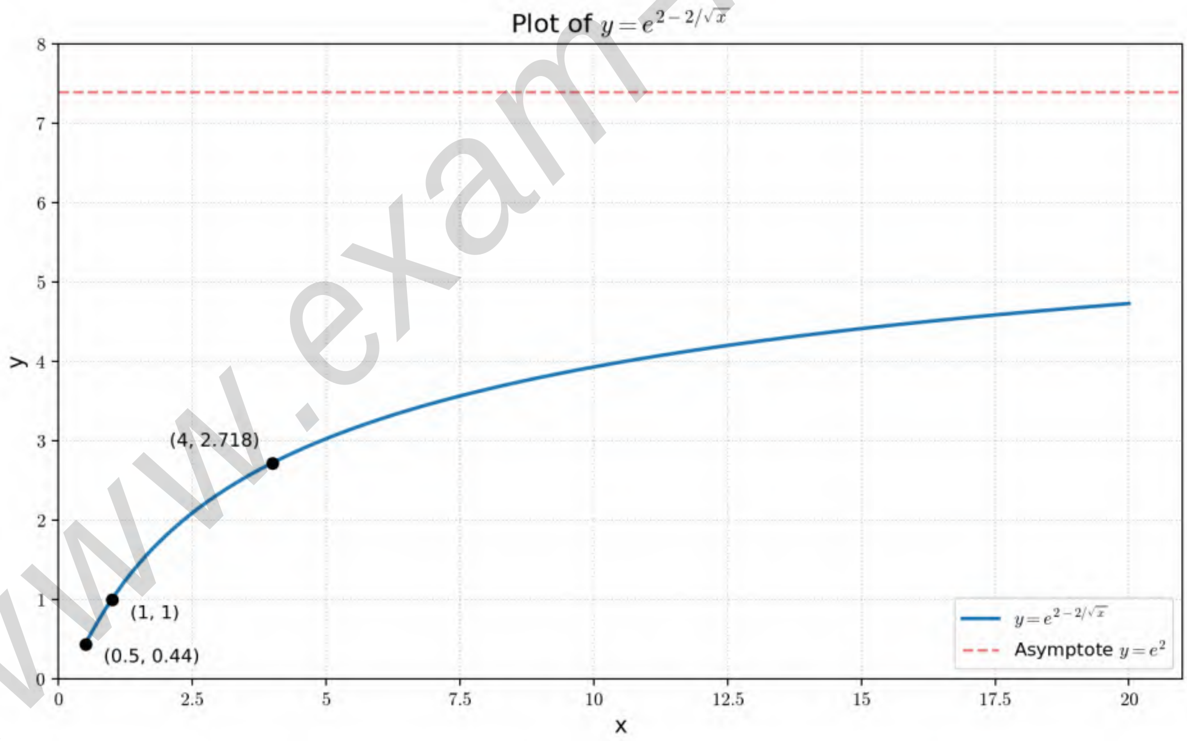
Substitute $A = -2$ and $C = 2$ back into the logarithmic equation:

$$\ln y = \frac{-2}{\sqrt{x}} + 2$$

To express y in terms of x, take the exponential of both sides:

$$y = e^{2-\frac{2}{\sqrt{x}}}$$

$$y = e^{2(1-\frac{1}{\sqrt{x}})}$$



5. Behavior as x Tends to Infinity

We analyze the **limit** of the function as $x \to \infty$:

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{2-\frac{2}{\sqrt{x}}}$$

$$\text{As}\quad x \to \infty, \frac{2}{\sqrt{x}} \to 0$$

$$\lim_{x\to\infty} y = e^{2-0} = e^2$$

As x tends to infinity, y approaches the constant value e^2.

Final Answers:

(a) The equation of the curve is:

$$\boxed{y = e^{2-\frac{2}{\sqrt{x}}}}$$

(b) As $x \to \infty$, y approaches e^2.

9709_31_Summer_2020_Q9

Solution

1. Show that angle ABC is a right angle

To show that $\angle ABC$ is a right angle, we must demonstrate that the vectors forming the sides meeting at vertex B are orthogonal. We define the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$ using the given **position vectors**:

- $\overrightarrow{OA} = 2\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}$
- $\overrightarrow{OB} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
- $\overrightarrow{OC} = 1\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}$

The vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are calculated as follows:

$$\begin{aligned}\overrightarrow{BA} &= \overrightarrow{OA} - \overrightarrow{OB} \\ &= (2-3)\mathbf{i} + (0-2)\mathbf{j} + (5-3)\mathbf{k} \\ &= -1\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \\ \overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB} \\ &= (1-3)\mathbf{i} + (1-2)\mathbf{j} + (1-3)\mathbf{k} \\ &= -2\mathbf{i} - 1\mathbf{j} - 2\mathbf{k}\end{aligned}$$

We use the **scalar product** (dot product) to check for orthogonality:

$$\begin{aligned}\overrightarrow{BA} \cdot \overrightarrow{BC} &= (-1)(-2) + (-2)(-1) + (2)(-2) \\ &= 2 + 2 - 4 \\ &= 0\end{aligned}$$

Since the scalar product is zero, the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are perpendicular. Thus, $\angle ABC = 90^\circ$.

2. Show that triangle ABC is isosceles

A triangle is **isosceles** if at least two of its sides have equal length. We calculate the magnitudes of the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$:

$$\begin{aligned}|\overrightarrow{BA}| &= \sqrt{(-1)^2 + (-2)^2 + 2^2} \\ &= \sqrt{1+4+4} \\ &= \sqrt{9} = 3 \\ |\overrightarrow{BC}| &= \sqrt{(-2)^2 + (-1)^2 + (-2)^2} \\ &= \sqrt{4+1+4} \\ &= \sqrt{9} = 3\end{aligned}$$

Since $|\overrightarrow{BA}| = |\overrightarrow{BC}| = 3$, the triangle ABC is isosceles.

3. Find the exact length of the perpendicular from O to the line through B and C

The line L passing through B and C can be expressed in vector form as:

$$\mathbf{r} = \overrightarrow{OB} + \lambda(\overrightarrow{OC} - \overrightarrow{OB})$$

Using $\overrightarrow{BC} = -2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ as the direction vector $\mathbf{d}$:

$$\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(-2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

Let P be the foot of the perpendicular from the origin O to the line. The position vector of P is:

$$\overrightarrow{OP} = (3 - 2\lambda)\mathbf{i} + (2 - \lambda)\mathbf{j} + (3 - 2\lambda)\mathbf{k}$$

Since $\overrightarrow{OP}$ is perpendicular to the line, its scalar product with the direction vector $\mathbf{d}$ must be zero:

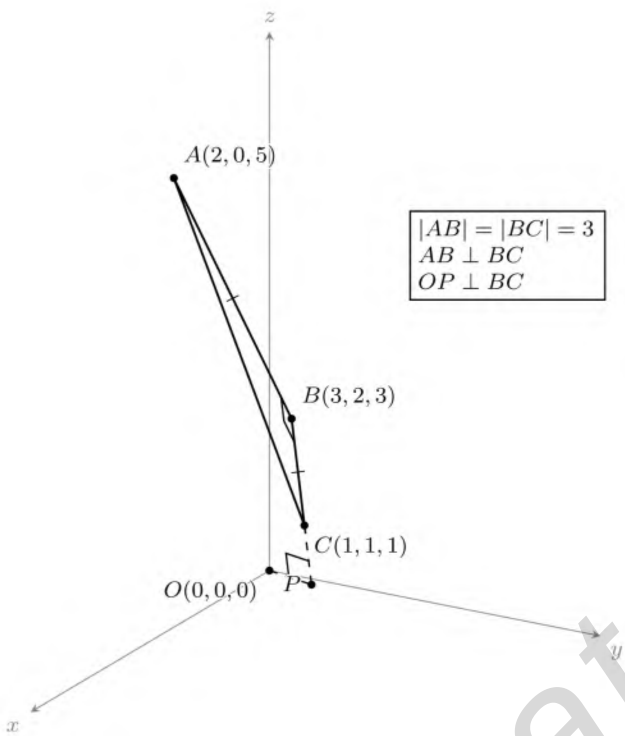
$$\begin{aligned} \overrightarrow{OP} \cdot \mathbf{d} &= 0 \\ (3 - 2\lambda)(-2) + (2 - \lambda)(-1) + (3 - 2\lambda)(-2) &= 0 \\ -6 + 4\lambda - 2 + \lambda - 6 + 4\lambda &= 0 \\ 9\lambda - 14 &= 0 \\ \lambda &= \frac{14}{9} \end{aligned}$$

Substitute $\lambda = \frac{14}{9}$ back into the expression for $\overrightarrow{OP}$:

$$\begin{aligned} \overrightarrow{OP} &= \left(3 - 2\left(\frac{14}{9}\right)\right)\mathbf{i} + \left(2 - \frac{14}{9}\right)\mathbf{j} + \left(3 - 2\left(\frac{14}{9}\right)\right)\mathbf{k} \\ &= \left(\frac{27 - 28}{9}\right)\mathbf{i} + \left(\frac{18 - 14}{9}\right)\mathbf{j} + \left(\frac{27 - 28}{9}\right)\mathbf{k} \\ &= -\frac{1}{9}\mathbf{i} + \frac{4}{9}\mathbf{j} - \frac{1}{9}\mathbf{k} \end{aligned}$$

The length of the perpendicular is the magnitude $|\ \overrightarrow{OP}\ |$:

$$\begin{aligned} |\ \overrightarrow{OP}\ | &= \sqrt{\left(-\frac{1}{9}\right)^2 + \left(\frac{4}{9}\right)^2 + \left(-\frac{1}{9}\right)^2} \\ &= \sqrt{\frac{1}{81} + \frac{16}{81} + \frac{1}{81}} \\ &= \sqrt{\frac{18}{81}} \\ &= \sqrt{\frac{2}{9}} \\ &= \frac{\sqrt{2}}{3} \end{aligned}$$



$$\boxed{\frac{\sqrt{2}}{3}}$$

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Solution

1. Expressing u in Cartesian form

To express the **complex number** $u = \frac{3i}{a+2i}$ in the form $x + iy$, we multiply the numerator and the denominator by the **complex conjugate** of the denominator, which is $a - 2i$.

$$\begin{aligned} u &= \frac{3i}{a+2i} \cdot \frac{a-2i}{a-2i} \\ &= \frac{3ai - 6i^2}{a^2 - (2i)^2} \\ &= \frac{3ai - 6(-1)}{a^2 - 4(-1)} \\ &= \frac{6 + 3ai}{a^2 + 4} \end{aligned}$$

Separating the real and imaginary parts:

$$u = \frac{6}{a^2+4} + i\left(\frac{3a}{a^2+4}\right)$$

Thus, the **Cartesian form** is:

$$\boxed{u = \frac{6}{a^2+4} + i\frac{3a}{a^2+4}}$$

2. Finding the value of a for $\arg u^* = \frac{1}{3}\pi$

- **Step 1: Relate $\arg u^*$ to $\arg u$.** Using the properties of the **argument** of a complex number, we know that for any complex number z, $\arg z^* = -\arg z$. Given $\arg u^* = \frac{1}{3}\pi$, it follows that:

$$\arg u = -\frac{1}{3}\pi$$

- **Step 2: Express $\arg u$ in terms of a.** From the Cartesian form $u = x + iy$ derived in part (i), we have:

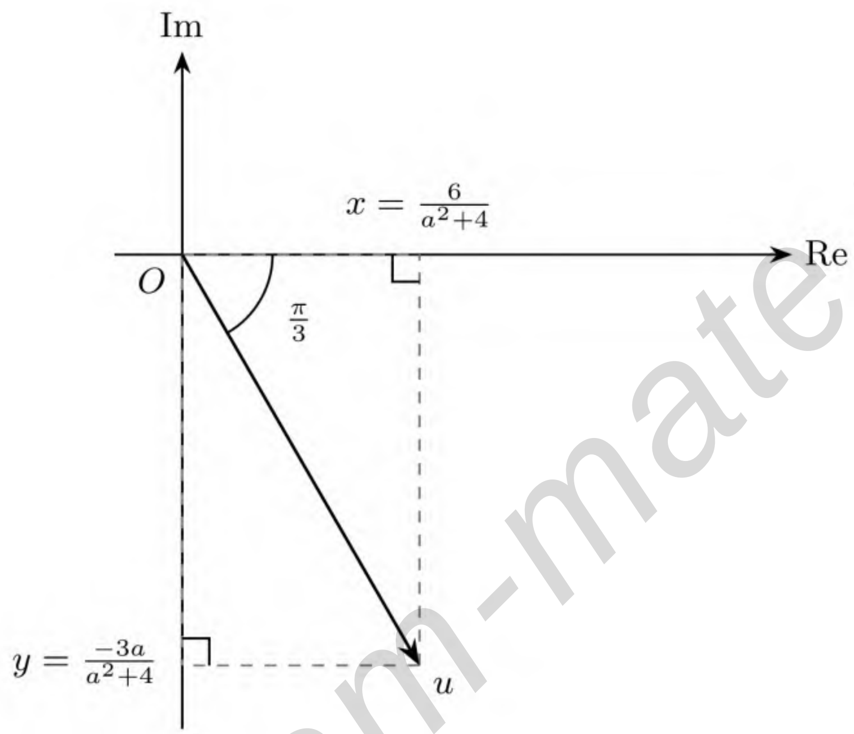
$$x = \frac{6}{a^2+4}, \quad y = \frac{3a}{a^2+4}$$

  Since $a^2 + 4 > 0$ for all real a, the real part x is always positive. This means u lies in the first or fourth quadrant of the **Argand diagram**. The argument is given by:

$$\tan(\arg u) = \frac{y}{x}$$

- **Step 3: Solve for a.** Substitute the expressions for x and y:

$$\tan\left(-\frac{1}{3}\pi\right) = \frac{\frac{3a}{a^2+4}}{\frac{6}{a^2+4}}$$
$$-\sqrt{3} = \frac{3a}{6}$$
$$-\sqrt{3} = \frac{a}{2}$$
$$a = -2\sqrt{3}$$



The exact value of a is:

$$\boxed{a = -2\sqrt{3}}$$

2: Argand diagram and complex inequalities

Solution

1. Analysis of the Inequalities

The region in the **Argand diagram** is defined by the intersection of two complex inequalities.

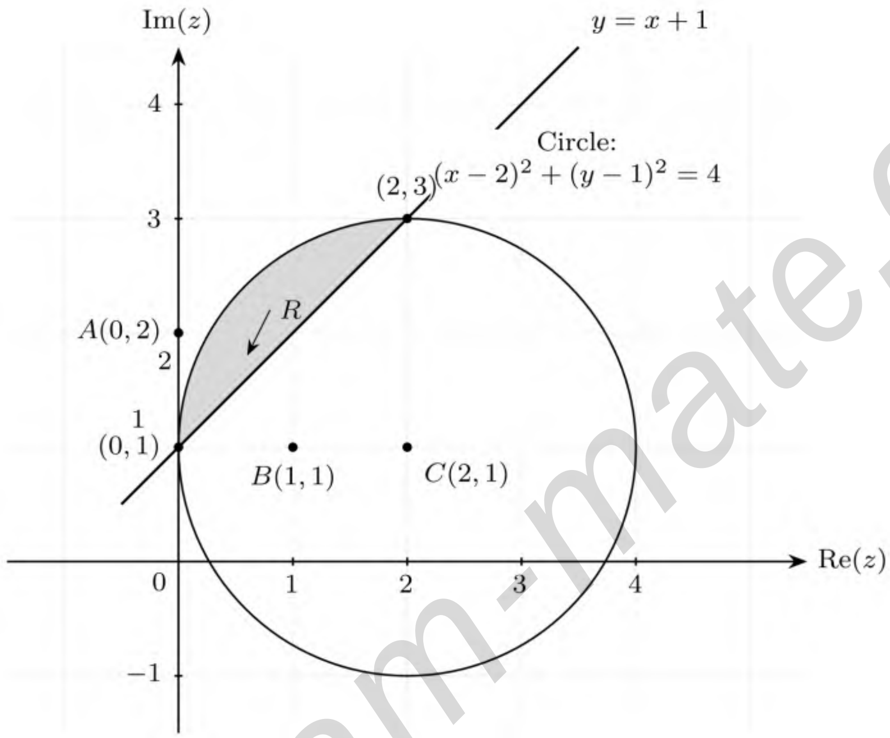
- **First Inequality:** $|\ z - 2i\ | \leq |\ z - (1+i)\ |$ This represents the set of points z whose distance from $2i$ is less than or equal to their distance from $1+i$. The boundary $|\ z - 2i\ | = |\ z - (1+i)\ |$ is the **perpendicular bisector** of the line segment joining the points $A(0,2)$ and $B(1,1)$.
  - Midpoint of AB: $M = \left(\frac{0+1}{2}, \frac{2+1}{2}\right) = (0.5, 1.5)$.
  - Gradient of AB: $m_{AB} = \frac{1-2}{1-0} = -1$.
  - Gradient of the bisector: $m_{\perp} = -\frac{1}{-1} = 1$.
  - Equation of the boundary line: $y - 1.5 = 1(x - 0.5) \implies y = x + 1$.

- Testing a point: For $z = 0$ (the origin), $|\ 0 - 2i\ | = 2$ and $|\ 0 - (1 + i)\ | = \sqrt{2}$. Since $2 \not\leq \sqrt{2}$, the origin is not in the region. The region is the half-plane **above** or on the line $y = x + 1$.

- **Second Inequality:** $|\ z - (2 + i)\ | \leq 2$ This represents the interior and boundary of a **circle** centered at $C(2, 1)$ with a radius $r = 2$.

2. Sketching the Region

The region is the intersection of the disk centered at $(2, 1)$ with radius 2 and the half-plane defined by $y \geq x + 1$.



3. Calculating the Least Value of $\arg z$

The **argument** $\arg z$ represents the angle the vector z makes with the positive real axis. To find the least value of $\arg z$ in the shaded region, we look for the point in the region that forms the smallest angle with the positive real axis.

- The shaded region is bounded by the arc of the circle $|\ z - (2 + i)\ | = 2$ and the chord $y = x + 1$.

- The intersection points of the line $y = x + 1$ and the circle $(x - 2)^2 + (y - 1)^2 = 4$ are found by substitution:

$$
\begin{aligned}
(x-2)^2 + (x+1-1)^2 &= 4 \\
(x-2)^2 + x^2 &= 4 \\
x^2 - 4x + 4 + x^2 &= 4 \\
2x^2 - 4x &= 0 \\
2x(x-2) &= 0
\end{aligned}
$$

This gives $x = 0$ (where $y = 1$) and $x = 2$ (where $y = 3$). The intersection points are $P_1(0, 1)$ and $P_2(2, 3)$.

- The region consists of points $z = x + iy$ such that $(x - 2)^2 + (y - 1)^2 \leq 4$ and $y \geq x + 1$.

- For any point z in the first quadrant, $\arg z = \arctan(y/x)$. To minimize this, we want to minimize the slope of the line connecting the origin to z.
- Looking at the geometry, the point in the shaded region with the smallest angle is the intersection point $P_2(2, 3)$. Any other point in the shaded region (which lies “above” the line segment P_1P_2 or on the arc) will have a larger ratio of y/x.
- Specifically, at $P_1(0, 1)$, $\arg z = \pi/2 \approx 1.57\,\text{rad}$.
- At $P_2(2, 3)$, $\arg z = \arctan(3/2) \approx 0.983\,\text{rad}$.
- We must also check if a line from the origin can be **tangent** to the circle within the shaded arc. A tangent from the origin to the circle $(x-2)^2 + (y-1)^2 = 4$ would satisfy the condition that the distance from $(2, 1)$ to the line $y = mx$ is 2:

$$\begin{aligned} \frac{|\ m(2) - 1\ |}{\sqrt{m^2+1}} &= 2 \\ (2m-1)^2 &= 4(m^2+1) \\ 4m^2 - 4m + 1 &= 4m^2 + 4 \\ -4m &= 3 \\ m &= -0.75 \end{aligned}$$

  A negative slope corresponds to the fourth quadrant, which is outside our shaded region ($y \geq x + 1$). Thus, the minimum argument must occur at one of the “corners” of the region.

Comparing the arguments of the boundary points:

- For $z = 2 + 3i$: $\arg z = \arctan(1.5) \approx 0.9828\,\text{rad}$.
- For $z = 0 + i$: $\arg z = \pi/2 \approx 1.5708\,\text{rad}$.

The least value is at $z = 2 + 3i$.

$$\boxed{\arg z = \arctan(1.5) \approx 0.983\,\text{rad}}$$